Some Order Theory

Recall that a relation is a triple $R = (A, B, \mathrm{Gr}(R))$, where $\mathrm{Gr}(R) \subset A \times B$ is the graph of R. Whereas functions (a special case of a relation) may be interpreted as 'acting' on the elements of A and returning elements of B, in general it is fruitful to think of a relation R as 'relating' or 'ordering' elements of A and B, in the sense that $a \in A$ and $b \in B$ are **R-related** if $(a, b) \in \mathrm{Gr}\, R$.

As with functions (and, in fact, even more so in this case), we shall usually identify a relation R with its graph $\mathrm{Gr}\, R$.

We will be particularly interested in the case where $A = B$, and we shall say that $R = (A, A, \mathrm{Gr}\, R)$ is a **relation on** A. We call a pair (A, R) a **set equipped with a relation**, or an **ordered set**, if R is a relation on A. We call A the **domain** or **underlying set** of the ordered set (A, R), and R is called the **relation** or **ordering** of the ordered set (A, R). [It is worth mentioning that the pair (A, R) doesn't really provide any extra information than the relation R would already contain (since it already has its first two components equal to A), but we consider the pair if only to emphasize the fact that we wish to consider (A, R) as a set for which there is a relation acting between its elements.] When R is understood, we shall typically refer to (A, R) by the underlying set A (however, we shall be unusually careful). Denote by $\mathrm{Rel}(A)$ the set of all relations on A.

Unless otherwise mentioned, in this section 'relation' will be meant to mean a binary relation whose domain and codomain coincide. Given such a (binary) relation R on A, we use the infix notation xRy to mean $(x, y) \in \mathrm{Gr}\, R$. We shall also typically use symbols like $\sim, \simeq, \approx, \equiv, \cong, <, \leq, \prec, \preceq, \sqsubset, \sqsubseteq, \subset, \subseteq$, etc.

Example 1

Given any set A, we may define a relation R on A by $(x, y) \in \mathrm{Gr}\, R$ if and only if $x \subset y$. By abuse of notation, we denote this relation by $\subset$ as well and call it the **subset relation on** A.

Example 2

Likewise, given any set A, we may define a relation R on A by $(x, y) \in \mathrm{Gr}\, R$ if and only if $x \in y$. Again, by abuse of notation, we denote this relation by $\in$ as well and call it the **membership relation on** A.

Example 3

The standard ordering $\leq$ defines a relation R on $\mathbb{N}$. Specifically, $(n, m) \in \mathrm{Gr}\, R$ if and only if there is p such that $n + p = m$.

Example 4

Let $A = \{1, 2, 3, 4\}$. Then $\mathrm{Gr}\, R \coloneqq \{(1, 2), (1, 3), (1, 4), (2, 4), (3, 4)\}$ defines a relation on A.

Example 5

Let $A = \{1, 2, 3, 4, 5\}$. Then $\mathrm{Gr}\, R \coloneqq \{(1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 5), (4, 5)\}$ defines a relation on A.

Example 6

For any set A, we define a relation R on A by $(x, y) \in \operatorname{Gr} R$ if and only if $x = y$. By abuse of notation, we denote this relation by $=$ and call it the **equality relation on** A.

Example 7

For any set A, the sets $A \times A$ and $\varnothing$ define (graphs of) relations on A.

Example 8

For any set A, we define a relation R on A by $(x, y) \in \operatorname{Gr} R$ if and only if there exists a bijection $f : x \to y$. When $(x, y) \in \operatorname{Gr} R$, we write $|x| = |y|$, and say that x and y have the same cardinality (note that this is independent of the containing set A).

Just as with algebras, we are also interested in subobjects and of maps between relations on a set that respect the relations.

Suppose that we have two ordered sets (A, R) and (B, S). Then we say that (A, R) is a **ordered subset** of (B, S) if $A \subset B$ and for every $a, b \in A$, we have $(a, b) \in \operatorname{Gr} S$ if and only if $(a, b) \in \operatorname{Gr} S$. When (A, R) is an ordered subset of (B, S), we write $(A, R) \subset (B, S)$.

Naturally, given an ordered set (A, R) and a subset $B \subset A$, we might ask if it is possible to 'restrict' the relation R to B. Given a relation $R = (A, B, \operatorname{Gr} R)$, and subsets $C \subset A$ and $D \subset B$, we define the **restriction of** R **to** C, D as the relation

$$R|_{C,D} := (C, D, (C \times D) \cap \operatorname{Gr} R).$$

It is important to note that given an ordered set (A, R) and two ordered subsets (B, R') and (B, R''), it must be the case that $R' = R'' = R|_B$:

Lemma 0.1

Suppose that $(B, R') \subset (A, R)$. Then $R' = R|_B$.

Thus, an ordered subset is determined entirely by its underlying set.

Example 9

Suppose we have given the ordered set $(\mathcal{P}(S), \subset)$. Then take B to be the set of all singletons. Then $\operatorname{Gr} \subset|_B = \{(\{a\}, \{a\}) \mid a \in S\}$. On the other hand, let C be the set of all finite subsets of S. Then $\operatorname{Gr} \subset|_C = \{(x, y) \in \mathcal{P}(S) \times \mathcal{P}(S) \mid x \subset y \text{ and } x, y \text{ finite}\}$.

Suppose that (A, R) and (B, S) are two ordered sets. Then we define a **order-preserving map** $\varphi : (A, R) \to (B, S)$ to be a triple $((A, R), (B, S), \varphi : A \to B)$ (following the convention of functions in that the order-preserving map records the relevant objects) such that for every $a, b \in A$ we have that $(a, b) \in \operatorname{Gr} R$ implies $\varphi(a) = \varphi(b)$ or $(\varphi(a), \varphi(b)) \in \operatorname{Gr} S$, and the function $\varphi : A \to B$ is said to be a **order-preserving map from** (A, R) **into** (B, S). It is also common to call such a triple $((A, R), (B, S), \varphi)$ a **monotonic function** or to say that it is **monotonic**.

Example 10

The map $\varphi : (\mathbb{N}, \leq) \to (\mathbb{N}, \leq)$ defined by $\varphi(n) = \begin{cases} n - 1 & \text{if } n \geq 1 \\ 0 & \text{if } n = 0 \end{cases}$ defines an order-preserving function.

Example 11

The map $\varphi : (\mathbb{N}, \leq) \to (\mathcal{P}(\mathbb{N}), \subset)$ defined by $\varphi(n) = \{n\}$ is *not* an order-preserving function. However, the map $\psi : (\mathbb{N}, \leq) \to (\mathcal{P}(\mathbb{N}), \subset)$ defined by $\psi(n) = \{0, \ldots, n\}$ *is* an order-preserving function.

We say that an order-preserving map $\varphi : (A, R) \to (B, S)$ is an **order-embedding** if

(i) $\varphi : A \to B$ is injective and

(ii) for every $a, b \in A$ we have $(a, b) \in \operatorname{Gr} R$ *if and only if* $(\varphi(a), \varphi(b)) \in \operatorname{Gr} S$.

An **order-isomorphism** is a bijective order-embedding.

Example 12

The map ψ in **??** defines an order-embedding.

Being injective (or even bijective) alone does not imply that an order-preserving map is an order-embedding:

Example 13

Suppose $B \subsetneq A$, and (A, R) is a ordered set. Then the identity map id_A defines an order-preserving map $\mathrm{id}_A : (A, R|_B) \to (A, R)$. If there is at least one pair $(a, b) \in \operatorname{Gr} R$ such that at least one of a and b lie in $A \smallsetminus B$ (and so $(a, b) \notin \operatorname{Gr} R|_B$), then this will not be an order-embedding.

Naturally, we may show that the composition of two order-preserving maps is an order-preserving map:

Lemma 0.2

Suppose $\varphi : (A, R) \to (B, S)$ and $\psi : (B, S) \to (C, T)$ are order-preserving maps. Then $\psi \circ \varphi : (A, R) \to (C, T)$ is an order-preserving map. Moreover, if φ and ψ are order-embeddings, then $\psi \circ \varphi$ is an order-embedding.

Section 1

Strict Partial Orders and Partial Orders

Our second interpretation of a relation is that of *containment* or of ordering by *size*. The motivating examples in this case are the subset relation $\subset$ on $\mathbb{P}(S)$ for some set S, or the membership relation $\in$ on a set S. However, in general the membership relation $\in$ on a set does not have many nice properties, so our main motivation shall be $\subset$ on $\mathcal{P}(S)$. But there is also $\subsetneq$, which shall serve as the basis for our first definition: an ordered set $(P, <)$ is said to be a **strict partially-ordered set**, or **strict poset**, if

(i) $<$ is irreflexive, i.e. $x < x$ is never true for $x \in P$, and

(ii) $<$ is transitive, i.e. $x < y$ and $y < z$ implies $x < z$ for every $x, y, z \in P$.

The relation $<$ is called a **strict partial order**. A strict partial order in general satisfies another property; it is *asymmetric*. We say that a relation R on a set A is **asymmetric** if $(a, b) \in \operatorname{Gr} R$ implies $(b, a) \notin \operatorname{Gr} R$. To see why $<$ is asymmetric, suppose for the sake of a contradiction that $x < y$ and $y < x$. Then by transitivity, we have $x < x$, from which we reach a contradiction by the fact that $<$ is irreflexive.

What of $\subset$, then? The relation between $\subset$ and $\subsetneq$ is quite simple: $A \subset B$ if and only if $A \subsetneq B$ or $A = B$. For this reason, it seems worthwhile to consider, given a strict partial order $<$, the new relation $<^=$, which we denote by $\le$, defined by $x \le y$ if and only if $x < y$ or $x = y$. Alternatively, it is the relation defined by

$$(\mathrm{Gr} \le) = (\mathrm{Gr} <) \cup (\mathrm{Gr} =).$$

Let us examine the properties of $\le$:

(i) $\le$ is reflexive, which follows by definition.

(ii) $\le$ is transitive, as given $x \le y$ and $y \le z$, we have several cases:

Case 1: $x < y$ and $y < z$. In this case, that $x < z$ and hence $x \le z$ follows from the fact that $<$ is transitive by hypothesis.

Case 2: $x = y$ and $y < z$. In this case, $x < z$ by assumption, substituting y for x.

Case 3: $x < y$ and $y = z$. Yet again, we have $x < z$ by assumption, substituting y for z.

Case 4: $x = y$ and $y = z$. In this case, $x = z$ by the transitivity of $=$.

(iii) Given that $<$ is asymmetric, we might ask what sort of implications exist when $x \le y$ and $y \le x$. Indeed, because of the fact that $<$ is irreflexive and asymmetric, it must be that $x = y$. Otherwise, we have $x < y$ and $y < x$, which is not possible. A relation R for which $(x, y), (y, x) \in \mathrm{Gr}\, R$ implies $x = y$ is said to be **anti-symmetric**. In effect, it is a relation in which there are no 'non-trivial' symmetries.

We say that an ordered set $(P, \le)$ is a **partially-ordered set**, or a **poset**, if

(i) $\le$ is reflexive,

(ii) $\le$ is anti-symmetric, and

(iii) $\le$ is transitive.

The relation $\le$ is called a **partial order**. Sometimes we call a poset $(P, \le)$ a **non-strict poset** and $\le$ a **non-strict partial order**.

In the same way that $A \subset B$ if and only if $A \subsetneq B$ or $A = B$, we could also describe $\subsetneq$ in terms of $\subset$: $A \subsetneq B$ if and only if $A \subset B$ and $A \ne B$. Thus, given a partial order $\le$, we could ask whether the relation $\le^{\ne}$, which we denote by $<$, defined by $x \le y$ and $x \ne y$, or

$$(\mathrm{Gr} <) = (\mathrm{Gr} \le) \setminus (\mathrm{Gr} =),$$

is a strict partial order. And indeed it is:

(i) $<$ is irreflexive, as if $x < x$, then by definition $x \le x$ and $x \ne x$. But $x \ne x$ never holds, so it must be that $x < x$ never holds.

(ii) $<$ is transitive, as if $x < y$ and $y < z$, then $x \le y$ and $y \le z$ in particular, so $x \le z$ by the fact that $\le$ is transitive. Now, if $x = z$, then we would have $x \le y$ and $y \le x$, so that by anti-symmetry of $\le$ we find that $x = y$. But by the definition of $<$, we know that $x \ne y$, leading to a contradiction. Hence, $x \le z$ and $x \ne z$, so $x < z$.

We thus find that partial orders and strict partial orders are related to one-another in a very strong way, and for this reason we shall freely use both the strict and non-strict versions of a partial order when dealing with either strict posets or (non-strict) posets, which in the above sense are 'equivalent'.

In general, given a poset (P, R), we denote the strict partial order associated with R by $R^{\ne}$, whereas given a strict poset (P, R), we denote the (non-strict) partial order associated with R by $R^{=}$. When possible, we shall adopt the convention that the (non-strict) partial order associated with a strict partial order be represented by adding a bar beneath the symbol of the strict partial order, at least when possible, such as with $<$ and $\le$ or $\prec$ and $\preceq$. However, there are exceptions, such as $\subsetneq$ and $\subset$.

Example 14

$(\mathcal{P}(S), \subset)$ is a poset, while $(\mathcal{P}(S), \subsetneq)$ is the associated strict poset.

Example 15

$(\mathbb{N}, \leq)$ is a poset, while $(\mathbb{N}, <)$ is the associated strict poset.
The algebraic structure of $\mathbb{N}$ behaves well with respect to $\mathbb{N}$. Indeed, given $n \leq m$ and any p, we find that $n \cdot p \leq m \cdot p$ since there exists an injection of $n \times p$ into $m \times p$ given by $\iota \times \mathrm{id}_p$, with $\iota : n \to m$ the inclusion of n into m. Likewise, $n + p \leq n + q$, again by the fact that there exists an injection of $n \amalg p$ into $m \amalg p$ given by $\iota \amalg \mathrm{id}_p$. We say that $+$ and $\cdot$ are **compatible with** $\leq$.
We can strengthen the above identities; if $n < m$ and $p \neq 0$, then $n \cdot p < m \cdot p$, with the same reasoning: there is an injection of $n \times p$ into $m \times p$ that is not a surjection, showing that $n \cdot p < m \cdot p$. Additionally, with $n < m$ and p arbitrary, we also find that $n + p < m + p$, again by the same reasoning as above.
With this in mind, we have **cancellation laws** for $+$ and $\cdot$; if $n + p = m + p$, then $n = m$, and if $n \cdot p = m \cdot p$, then either $n = m$ or $p = 0$.

Example 16

Define the relation $\mid$ on $\mathbb{N}$, called the **divibility relation**, by $n \mid m$ if and only if there exists $p \in \mathbb{N}$ such that $n \cdot p = m$.
$\mid$ is reflexive, as $n \cdot 1 = n$. It is anti-symmetric, as if $n \mid m$ and $m \mid n$, then there exists p and q such that $n \cdot p = m$ and $m \cdot q = n$. Thus, we have $n = (n \cdot p) \cdot q = n \cdot (p \cdot q)$ and $m = m \cdot (p \cdot q)$. So either $n \neq 0$ and so $p \cdot q = 1$, which shows $p = q = 1$, $m \neq 0$, which also shows that $p = q = 1$, or $m = n = 0$. In any case, $n = m$. Finally, $\mid$ is transitive, as if $p \mid q$ and $q \mid r$ there are natural numbers n and m such that $p \cdot n = q$ and $q \cdot m = r$. But then $p \cdot (n \cdot m) = r$, so $p \mid r$.
Thus, we see that $(\mathbb{N}, \mid)$ is a poset.

Example 17

Given two partially-ordered sets $(P_1, \leq_1)$ and $(P_2, \leq_2)$, we may define a partial order $\leq$ on $P_1 \times P_2$ as follows: $(a, b) \leq (c, d)$ if and only if $a <_1 c$ or $a = c$ and $b \leq_2 d$.
The partial order $\leq$ is called the **lexicographical ordering** induced by $\leq_1$ and $\leq_2$, so named because it mirrors how words are alphabetized, and $(P_1 \times P_2, \leq)$ is called the **lexicographical product** of $(P_1, \leq_1)$ and $(P_2, \leq_2)$.

Example 18

Given two partially-ordered sets $(P_1, \leq_1)$ and $(P_2, \leq_2)$, we may define a partial order $\leq$ on $P_1 \amalg P_2$ as follows: $(x, i) \leq (y, j)$ if and only if $i < j$, $i = j = 0$ and $x \leq_1 y$, or $i = j = 1$ and $x \leq_2 y$.
The poset $(P_1 \amalg P_2, \leq)$ is called the **ordinal sum** of $(P_1, \leq_1)$ and $(P_2, \leq_2)$, and is denoted by $(P_1, \leq_1) \oplus (P_2, \leq_2)$.
It is important to note that $(P_1, \leq_1)$ and $(P_2, \leq_2)$ both embed into $(P_1, \leq_1) \oplus (P_2, \leq_2)$ via the canonical inclusions $\iota_1 : P_1 \to P_1 \amalg P_2$ and $\iota_2 : P_2 \to P_1 \amalg P_2$.

Given an ordered-set (A, R), we may define another partial order on A which is 'dual' to R, which we denote by R^{op} and define by

$$\mathrm{Gr}\, R^{\mathrm{op}} := \{(a, b) \in A \times A \mid (b, a) \in \mathrm{Gr}\, R\}.$$

We call this new relation the **opposite relation** (or **opposite**, **dual**, **inverse**, etc) of R. For example, the opposite relation of $\subset$ on $\mathcal{P}(S)$ is $\supset$; $A \subset B$ if and only if $B \supset A$.

Another common notation, particularly when the symbol being used for the relation R is not symmetric about a vertical axis, is to flip the symbol along the vertical axis, as in $\subset$ and $\supset$ or $\leq$ and $\geq$. We shall adopt this convention when convenient and using $\leq^{\mathrm{op}}$ when there could be possible confusion. While $a \leq b$ may be read as ‘a is less than or equal to b’, we read $a \geq b$ as ‘a is greater than or equal to b’.

The importance of the opposite relation is that it captures a sense of ‘duality’. For example, given some property P of R, the ‘dual property’ P^{op} is a property of R^{op}, where every instance of R in P is replaced by R^{op}. A property P is said to be **self-dual** if P holds if and only if P^{op}. For example, reflexitivity is self-dual; R is reflexive if and only if $\leq^{\mathrm{op}}$ is. Likewise, anti-symmetry, transitivity, irreflexitivity, asymmetry, and symmetry are all self-dual properties:

Lemma 1.1

Let (A, R) be an ordered-set, and R^{op} the opposite relation of R.

(a) $(R^{\mathrm{op}})^{\mathrm{op}} = R$.

(b) R is reflexive if and only if R^{op} is.

(c) R is irreflexive if and only if R^{op} is.

(d) R is symmetric if and only if R^{op} is, and this is the case if and only if $R = R^{\mathrm{op}}$.

(e) R is asymmetric if and only if R^{op} is.

(f) R is anti-symmetric if and only if R^{op} is.

(g) R is transitive if and only if R^{op} is.

Corollary 1.2

If $(P, \leq)$ is a (non-strict or strict) poset, then $(P, \geq)$ is a (non-strict or strict) poset.

Section 2

Total Orders

As the example of $\mathcal{P}(S)$ makes apparent, it need not be the case that given two elements x and y of a poset we need have either $x \leq y$ or $y \leq x$. That is, not every pair of elements needs to be *comparable*. We can define two relations on the underlying set of a poset $(P, \leq)$; we define the **comparability relation**, $\perp$, by

$$(\mathrm{Gr} \perp) = \{(x, y) \mid x \leq y \text{ or } y \leq x\}.$$

We can immediately see that $\perp$ is reflexive (for $x \leq x$), symmetric (for the condition ‘$x \leq y$ or $y \leq x$’ is symmetric), but it need not be transitive, for $\{0,1\}, \{0,2\} \subset \{0,1,2\}$ but neither $\{0,1\} \subset \{0,2\}$ nor $\{0,2\} \subset \{0,1\}$. Similarly, we define the **incomparability relation**, $\parallel$, by

$$(\mathrm{Gr} \parallel) = (P \times P) \smallsetminus (\mathrm{Gr} \perp) = \{(x, y) \mid \text{neither } x \leq y \text{ nor } y \leq x\}.$$

$\parallel$ is irreflexive (since $\perp$ is reflexive) and symmetric (since $\perp$ is symmetric), but it also need not be transitive, for $\{0,1\} \parallel \{0,2\}$ and $\{0,2\} \parallel \{0,1,3\}$, but $\{0,1\} \perp \{0,1,3\}$.

When a partial order $\leq$ is such that $x \perp y$ for every pair $(x, y) \in P^2$, we say that $\leq$ is a **total order** and that $(P, \leq)$ is a **totally ordered set** (sometimes called a **linear order**). More generally, we say that a relation R on a set A is **total** if for every $(x, y) \in A^2$ either $(x, y) \in R$ or $(y, x) \in R$.

The ‘strict’ version of totality is known as *trichotomy*: a relation R on a set A is **trichotomous** if for every $(x, y) \in A^2$, exactly one of $x = y$, $x < y$, or $y < x$ occurs. Indeed, suppose that $\leq$ is a total order, and let $<$ be the associative strict partial order. Given x, y, we know that either $x \leq y$ or $y \leq x$. In terms of $<$, this means that either $x = y$ or $x < y$, or $x = y$ or $x = y$. That is, $x = y$, $x < y$, or $y < x$. That exactly one of these can happen follows from the fact that if $x = y$ and $x < y$, then $x < x$, leading to a contradiction. The same argument shows that $x = y$ and $y < x$ cannot occur. Finally, $x < y$ and $y < x$ cannot occur by the fact that $<$ is asymmetric.

EXAMPLE 19

$(\mathbb{N}, \leq)$ is a totally-ordered set, as is every (Von Neumann) natural number.

EXAMPLE 20

Let $(P_1, \leq_1)$ and $(P_2, \leq_2)$ be totally-ordered sets. We can show that $(P_1 \times p_2, \leq)$ with $\leq$ the dictionary order induced by $\leq_1$ and $\leq_2$ is a total order.

EXAMPLE 21

Let $(P_1, \leq_1)$ and $(P_2, \leq_2)$ be totally-ordered sets. We can show that the disjoint union $(P_1 \sqcup P_2, \leq) = (P_1, \leq_1) \sqcup (P_2, \leq_2)$ is also a total order.

SUBSECTION 2.1

ORDERED SUBSETS, ORDER-PRESERVING MAPS, AND POSETS

Concerning ordered subsets, we wish to show that the poset $(P, \leq)$ is an ordered subset of the poset $(Q, \leq)$ if and only if $(P, <)$ is an ordered subset of $(Q, <)$.

PROPOSITION 2.1

Let $(P, \leq)$ and $(Q, \leq)$ be posets and $(P, <)$ and $(Q, <)$ their respective strict versions. Then $(P, \leq)$ is an ordered subset of $(Q, \leq)$ if and only if $(P, <)$ is an ordered subset of $(Q, <)$.

We may also show that an ordered subset of a poset $(Q, \leq)$ is itself a poset, and in fact that being an ordered subset implies inheritance of whatever properties (among those we've seen thus far) the containing relation has:

PROPOSITION 2.2

Let (A, R) be an ordered subset of (B, S). Then

(a) if S is reflexive, then R is reflexive,

(b) if S is irreflexive, then R is irreflexive,

(c) if S is symmetric, then R is symmetric,

(d) if S is asymmetric, then R is asymmetric,

(e) if S is anti-symmetric, then R is anti-symmetric,

(f) if S is transitive, then R is transitive,

(g) if S is total, then R is total, and

(h) if S is trichotomous, then R is trichotomous.

COROLLARY 2.3

If (A, R) is an ordered subset of (B, S), then if (B, S) is a poset or a strict poset, then (A, R) is a poset or strict poset, respectively.

For this reason, we might call an ordered subset (A, R) or a poset $(P, \leq)$ a **partially-ordered subset** or **subposet**. If $(P, \leq)$ is moreover a totally-ordered set, then we might call (A, R) a **totally-ordered subset**.

We can also extend **??** to show that order isomorphisms preserve these properties (as we should hope they do, for otherwise we'd need to ask ourselves if order isomorphisms are actually capturing the entirety of the structure):

PROPOSITION 2.4

Let (A, R) and (B, S) be order isomorphic. Then

(a) if S is reflexive, then R is reflexive,

(b) if S is irreflexive, then R is irreflexive,

(c) if S is symmetric, then R is symmetric,

(d) if S is asymmetric, then R is asymmetric,

(e) if S is anti-symmetric, then R is anti-symmetric,

(f) if S is transitive, then R is transitive,

(g) if S is total, then R is total, and

(h) if S is trichotomous, then R is trichotomous.

COROLLARY 2.5

If (A, R) is a poset or a totally-ordered set, and is order isomorphic to (B, S), then (B, S) is a poset or a totally-ordered set, respectively.

EXAMPLE 22

Let $(P_1, \leq_1)$ and $(P_2, \leq_2)$ be posets. We define the relation $\leq$ on $P_1 \times P_2$ by $(a, b) \leq (c, d)$ if and only if $b <_2 d$ or, $b = d$ and $a \leq_1 c$; we call this the **colexicographical order** induced by $\leq_1$ and $\leq_2$, and $(P_1 \times P_2, \leq)$ the **colexicographical product** of $(P_1, \leq_1)$ and $(P_2, \leq_2)$. This should look surprisingly familiar; it is the lexicographical order but with the roles of $\leq_1$ and $\leq_2$ interchanged. While we could easily adapt the proof that the lexicographical order is a partial order in order to prove that the colexicographical order is as well, we can more simply reduce the colexicographical order to that of the lexicographical order by establishing an order isomorphism.

To this effect, let $\preceq$ be the lexicographical order induced by $\leq_2$ and $\leq_1$ (note the order of $\leq_1$ and $\leq_2$ here!), and then define $\tau : P_2 \times P_1 \to P_1 \times P_2$ by $\tau : (b, a) \mapsto (a, b)$. We claim that $\tau : (P_2 \times P_1, \preceq) \to (P_1 \times P_2, \leq)$ is an order-isomorphism. Indeed, $(b, a) \preceq (d, c)$ is precisely the statement that $b <_2 d$ or, $b = d$ and $a \leq_1 c$, so τ is certainly order-preserving, and is also a bijection, completely the proof.

Another common way to make use of **??** is to take a bijection $\varphi : A \to B$, where (A, R) is an ordered set, and *declare* that φ is an order-isomorphism. That is, we define a relation S on B by $(a, b) \in \operatorname{Gr} S$ if and only if $(\varphi^{-1}(a), \varphi^{-1}(b)) \in \operatorname{Gr} R$. In such a case, we would say that φ *induces* the relation S on B.

Finally, we can look at some of the general properties that partially-ordered sets and order-preserving maps between them have.

The first is a simplified criterion for being an order-perserving map: whereas with general ordered-sets we needed to account for the fact that an order-preserving map could be such that multiple elements had the same image regardless of the relation, this issue does not arise for posets. Indeed, if $\varphi(a) = \varphi(b)$, then $\varphi(a) \le \varphi(b)$ and $\varphi(b) \le \varphi(a)$, so the relation is preserved regardless of the relation between a and b (if they are even comparable). Thus, for $(P_1, \le_1)$ and $(P_2, \le_2)$ posets, the map $\varphi : (P_1, \le_1) \to (P_2, \le_2)$ is order-preserving exactly when $a \le_1 b$ implies $\varphi(a) \le_2 \varphi(b)$.

Additionally, it is particularly simple to describe order-embeddings for totally-ordered sets:

Lemma 2.6

Suppose $(T, \le_1)$ is totally-ordered and $(P, \le_2)$ is any poset. Then $\varphi : (T, \le_1) \to (P, \le_2)$ is an order-embedding if and only if given $x, y \in T$ with $x <_1 y$, we have $\varphi(x) <_2 \varphi(y)$.

Section 3

Upper and Lower Bounds, Maximal and Minimal Elements, and Suprema and Infima

Given the fact that the ordering $\le$ on a poset $(P, \le)$ can be thought of as measuring relative 'largeness', it is natural to ask about the 'largest' element or, dually, the 'smallest' element. We can refine this to talk about elements that are 'larger' (respectively, 'smaller') than a set of elements, and whether or not there is a smallest (respectively, largest) element among these.

There are two ways we could interpret 'largest' element; on one hand, we could interpret 'p is the largest element' to mean that $q \le p$ holds for every element $q \in P$, but on the other hand, we could interpret 'largest' to mean that $p < q$ never holds for $q \in P$. Ultimately, these two notions are *not* equivalent; in the first case, it is necessarily the case that p is comparable to every element of P, whereas in the second case we need not have this. Indeed, consider the following example:

Example 23

Consider the poset $(\{0, 1, 2, 3, 4\}, \le)$, where

$$(\mathrm{Gr} \le) = \{(0,0), (1,1), (2,2), (3,3), (4,4), (1,2), (1,3), (3,4), (2,4)\}.$$

Then both 0 and 4 satisfy the property that there are no element of $\{0, 1, 2, 3, 4\}$ strictly greater than them, but the are not greater than every element of $\{0, 1, 2, 3, 4\}$.

Let (A, R) be an ordered set. We say that an element $a \in A$ for which $(b, a) \in \mathrm{Gr}\, R$ for every $b \in B$ is a **maximum** (with respect to R), whereas an element $a \in A$ for which there is no $b \ne a \in B$ with $(b, a) \in \mathrm{Gr}\, R$ is said to be a **maximal element**. Dually, an element $a \in A$ for which $(a, b) \in \mathrm{Gr}\, R$ for every $b \in A$ is a **minimum**, and an element $a \in A$ for which there is no $b \ne a \in A$ with $(a, b) \in \mathrm{Gr}\, R$ is a **minimal element**. As **??** points out, maximal elements need not be unique. However, maxima/minima (the plural for maximum and minimum, respectively) are unique, and is also a maximal/minimal element (and necessarily the only such maximal/minimal element):

Lemma 3.1

Let (A, R) be an ordered set for which R is anti-symmetric. If (A, R) has a maximum or minimum, then it is unique and is a maximal or minimal element, respectively.

Generalizing the notion of 'larger than' and 'less than' a bit, we can ask, given an ordered set (A, R) and a subset $B \subset A$, whether there are elements in A larger than every element of B. Likewise, we can ask if there are elements in A smaller than every element of B. We say that $a \in A$ such that $(b, a) \in \mathrm{Gr}\, R$ for every

$b \in B$ is a **upper bound of** B, and $a \in A$ such that $(a,b) \in \operatorname{Gr} R$ for every $b \in B$ is a **lower bound of** B. When there exists a *least* upper bound a of B, we call a the **least upper bound** of B or the **supremum** of B, and denote it by $\sup B$. Likewise, when there exists a *greatest* lower bound a of B, we call a the **least upper bound** of B or the **infimum** of B, and denote it by $\inf B$.

When $\sup B \in B$, we say that $\sup B$ is the **maximum element of** B, whereas if $\inf B \in B$, then we say that $\inf B$ is the **minimum element of** B. Indeed, if we were to consider $(B, R|_B)$, to say that $\sup B \in B$ is to say that there is an element of B such that $(b, \sup B) \in \operatorname{Gr} R$ for every $b \in B$, i.e. that $\sup B$ is a maximum of $(B, R|_B)$. Likewise, if $\inf B \in B$, then it is a minimum of $(B, R|_B)$. Note that unless R is anti-symmetric it may not be the case that the supremum or infimum of B is unique.

LEMMA 3.2

Let (A, R) be an ordered set.

(a) Suppose (B, S) is an ordered subset of (A, R), and $C \subset B$. Then if $a \in B$ is an upper bound, lower bound, maximum, or minimum of C relative to S, then a is an upper bound, lower bound, maximum, or minimum of C relative to R, respectively.

(b) Suppose $\varphi : (B, S) \to (A, R)$ is an order-isomorphism, and $C \subset B$. Then $b \in B$ is an upper bound, lower bound, maximum, minimum, supremum, or infimum of C if and only if $\varphi(b) \in A$ is an upper bound, lower bound, maximum, minimum, supremum, or infimum of $\varphi[C]$, respectively. $b \in B$ is a maximal element, minimal element, maximum, or minimum of B if and only if $\varphi(b) \in A$ is a maximal element, minimal element, maximum, or minimum of A, respectively.

We will be principally interested in the case where $(A, R) = (P, \leq)$ is a poset.

EXAMPLE 24

Consider $(\mathcal{P}(S), \subset)$. In this case, there is a maximum S, a minimum $\varnothing$, and given $Q \subset \mathcal{P}(S)$, both the supremum and infimum exist, and $\sup Q = \bigcup Q$ and $\inf Q = \bigcap Q$.

EXAMPLE 25

Consider $(\mathbb{N}, \leq)$. In this case, there is a minimum 0, but no maximum, for $n \leq n+1$ for every natural number n. Additionally, every non-empty subset of $\mathbb{N}$ has a minimum element. To see why, we can first show that every non-empty finite subset of $\mathbb{N}$ has both a minimum and maximum element:

LEMMA 3.3

Every non-empty finite subset of $\mathbb{N}$ has both a minimum and maximum element.

COROLLARY 3.4: WELL-ORDERING PRINCIPLE

Every non-empty subset of $\mathbb{N}$ has a minimum element.

EXAMPLE 26

Consider $(\mathbb{N}, |)$. In this case, there is a minimum 1 and a maximum 0.

Section 4

Hasse Diagrams

For small, finite, posets a useful tool in understanding and visualizing the poset is through the use of *Hasse diagrams*. The idea is that we may define a simple, directed acyclic graph D with $V(D) = P$ based on the partial order of a poset $(P, \leq)$ such that there is a directed walk (necessarily a path because the digraph is acyclic) from $v \in P$ to $w \in P$ if and only if $v < w$.

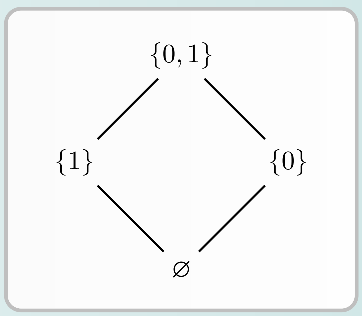
What we want to capture is the idea of an *immediate successor*. Let $(P, \leq)$ be any poset, and define the **covering relation** $\lessdot$ on P by $x \lessdot y$ if and only if $x < y$ and there does not exist z such that $x < z < y$. In this case, we call y the **immediate successor** of x, and x the **immediate predecessor** of y, and say that y **covers** x. $\lessdot$ is clearly irreflexive and asymmetric. However, it is not transitive, for if $x \lessdot y$ and $y \lessdot z$, then it cannot be the case that $x \lessdot z$, for although $x < z$ holds, we have $x < y < z$.

For finite posets, $\lessdot$ uniquely determines $<$ in exactly the way we wanted above.

Then we define a digraph D to be the digraph corresponding to $\lessdot$, i.e. with vertex set P and adjacency relation $\lessdot$. We call this diagraph the **Hasse diagram** of $(P, \leq)$. The graph drawings of D (also called Hasse diagrams) are the visual tools we're after. When drawing such graph drawings it is typical to use undirected edges, but ensuring that whenever (a, b) is an arc in D we draw the vertex corresponding to b above the vertex corresponding to a.

Example 27

Consider $(\mathcal{P}(\{0,1\}), \subset)$. In this case, we have $\varnothing \lessdot \{0\}$, $\varnothing \lessdot \{1\}$, $\{0\} \lessdot \{0,1\}$, and $\{1\} \lessdot \{0,1\}$. This gives us the Hasse diagram

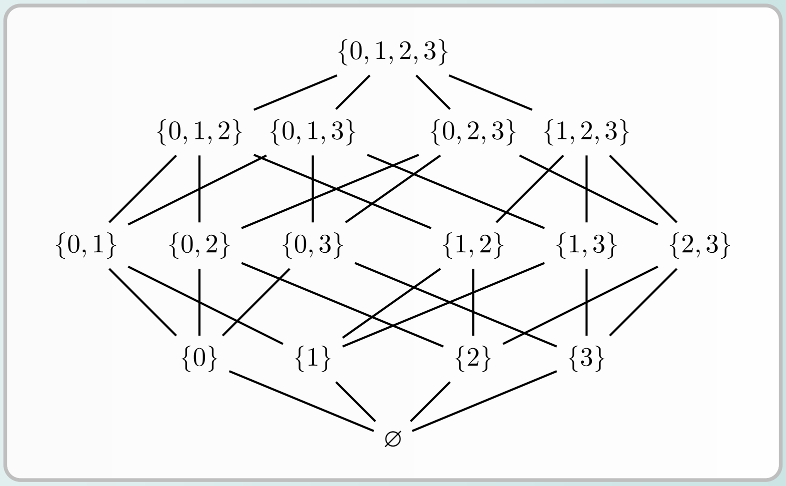


Example 28

Consider $(\mathcal{P}(\{0,1,2,3\}), \subset)$. In this case we have

$$\begin{aligned}\mathrm{Gr}(\lessdot) = \{&(\varnothing,\{0\}),(\varnothing,\{1\}),(\varnothing,\{2\}),(\varnothing,\{3\}),\\ &(\{0\},\{0,1\}),(\{0\},\{0,2\}),(\{0\},\{0,3\}),(\{1\},\{0,1\}),(\{1\},\{1,2\}),(\{1\},\{1,3\}),\\ &(\{2\},\{0,2\}),(\{2\},\{1,2\}),(\{2\},\{2,3\}),(\{3\},\{0,3\}),(\{3\},\{1,3\}),(\{3\},\{2,3\})\\ &(\{0,1\},\{0,1,2\}),(\{0,1\},\{0,1,3\}),(\{1,2\},\{0,1,2\}),(\{1,2\},\{1,2,3\}),\\ &(\{0,2\},\{0,1,2\}),(\{0,2\},\{0,2,3\}),(\{0,3\},\{0,1,3\}),(\{0,3\},\{0,2,3\}),\\ &(\{1,3\},\{0,1,3\}),(\{1,3\},\{1,2,3\}),(\{2,3\},\{0,2,3\}),(\{2,3\},\{1,2,3\}),\\ &(\{0,1,2\},\{0,1,2,3\}),(\{0,1,3\},\{0,1,2,3\}),(\{0,2,3\},\{0,1,2,3\}),(\{1,2,3\},\{0,1,2,3\})\}.\end{aligned}$$

This gives us the Hasse diagram



In general there are many different ways to draw a Hasse diagram. For example, instead we could have had